STEPS FOR SOLVING WORD PROBLEMS ON

MAXIMA AND MINIMA

Step 1: Figure with symbols for important variables.

Step 2: Write the objective function (the variable whose maxima or minima is sought) and the equation on constraints.

Step 3: Express the objective as a function of one variable using the equation on constraints.

Step 4: Determine the critical numbers (values of x for which derivative is zero).

Step 5: Analyze the critical numbers as maximum or minimum.

Step 6: Answer to the problem as posed.

MODEL ANSWERS

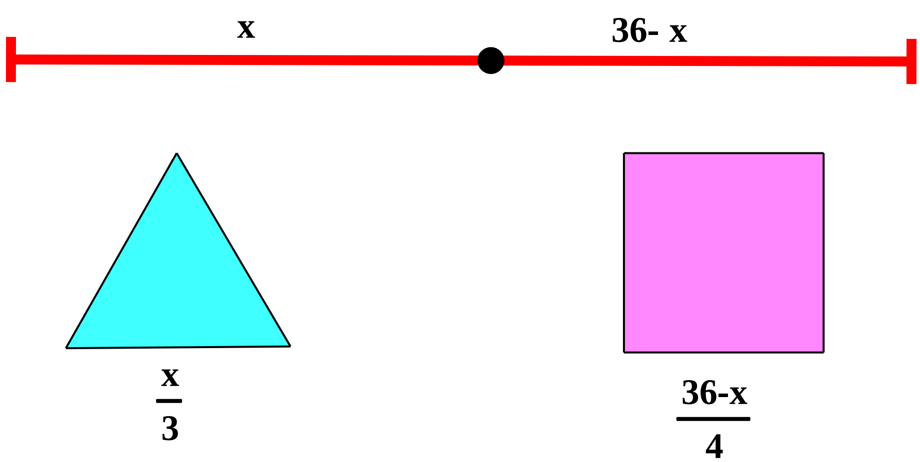
1. A wire of length 36 cm is cut into two pieces. One of the pieces is turned in the form of a square and the other in the form of an equilateral triangle. Find the length of

each piece so that the sum of the areas of the two is minimum.

SOLUTION:

Let the portion used to form the triangle be x m. The perimeter of the square will be

(36 – x) m.



Perimeter: $x = 3a$

Side: $a = \frac{x}{3}$

Area: $A = \frac{\sqrt{3}}{4} \times \left(\frac{x}{3}\right)^2$

Perimeter: $36 - x = 4s$

Length of side: $s = \frac{36 - x}{4}$

Area: $A = \left(\frac{36 - x}{4}\right)^2$

To minimize : $A(x) = \frac{\sqrt{3}}{36}x^2 + \left(\frac{36 - x}{4}\right)^2$, $\quad 0 \le x \le 36$

$$A'(x) = \frac{\sqrt{3}}{18}x + \left(\frac{36 - x}{4}\right)\left(\frac{-1}{2}\right)$$

For minimum, $A'(x) = 0$

$$\Rightarrow \frac{\sqrt{3}}{18}x + \frac{x}{8} = \frac{9}{2} \qquad \Rightarrow x = \frac{324}{9+4\sqrt{3}} \qquad A''(x) = \frac{\sqrt{3}}{18} + \frac{x}{8} > 0$$

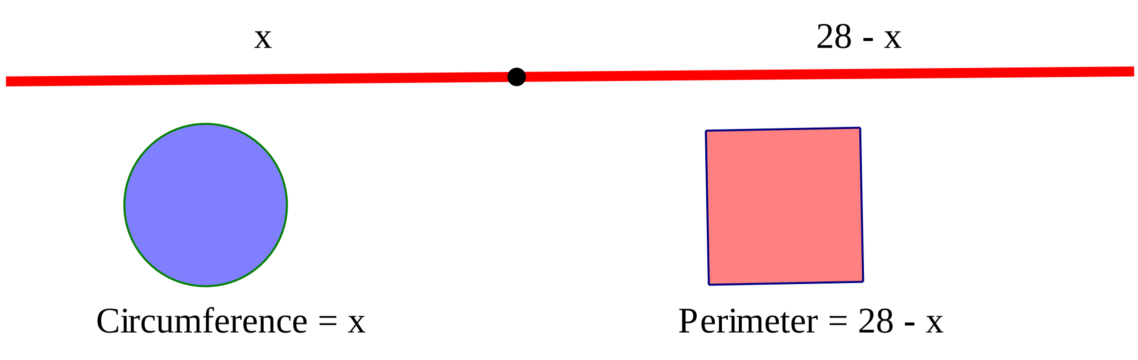
Therefore for minimum area, $\frac{324}{9+4\sqrt{3}}$ m must be cut to form the triangle and

remaining wire $\frac{144\sqrt{3}}{9+4\sqrt{3}}$ m is made into a square.

2. A wire of length 28 meters is to be cut into two pieces; one of the pieces is to be made into a square and the other into a circle. What should be the length of each piece so that the sum of the areas enclosed by them is minimum?

ANSWER:

Let the portions of the wire used to form the circle and the square have lengths x m and (28 – x) m respectively.



Circumference: $x = 2\pi r$

Perimeter: $28 - x = 4s$

Radius: $r = \frac{x}{2\pi}$

Length of side: $s = \frac{28-x}{4}$

Area: $A = \pi\left(\dfrac{x}{2\pi}\right)^2$

Area: $A = \left(\dfrac{28-x}{4}\right)^2$

To minimize: $f(x) = \pi\left(\dfrac{x}{2\pi}\right)^2 + \left(\dfrac{28-x}{4}\right)^2$, $0 \le x \le 28$

$$f'(x) = 2\pi\left(\frac{x}{2\pi}\right)\frac{1}{2\pi} + 2\left(\frac{28-x}{4}\right)\left(\frac{-1}{4}\right)$$

$$= \frac{x}{2\pi} + \frac{x-28}{8}$$

For minimum $f'(x) = 0$

$$4x + \pi x - 28\pi = 0$$

$$x = \frac{28\pi}{4+\pi}$$

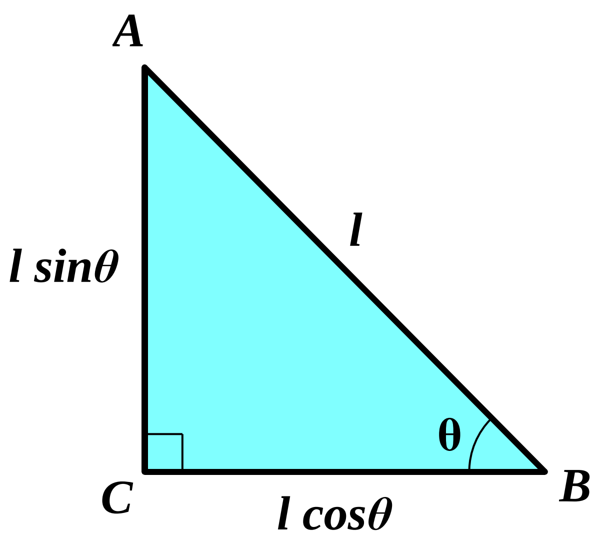
x	0	$\dfrac{28\pi}{4+\pi}$	28
$f(x)$	49	$\dfrac{196}{4+\pi}$	$\dfrac{196}{\pi}$

For minimum area $\dfrac{28\pi}{4+\pi}$ meters is to be made into a circle and $\dfrac{112}{4+\pi}$ meters is to be made into a square.

MAXIMA AND MINIMA

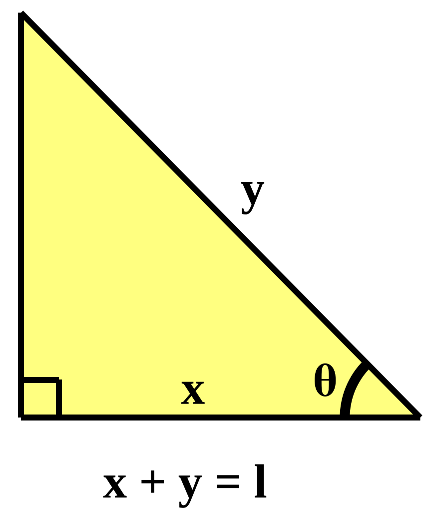
Problems defined on closed and bounded intervals

1. Prove that the area of a right-angled triangle of given hypotenuse is maximum when the triangle is isosceles.

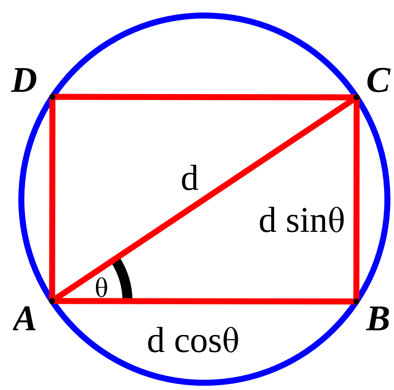


2. Find the largest possible area of a right-angled triangle whose hypotenuse is 5 cm long.

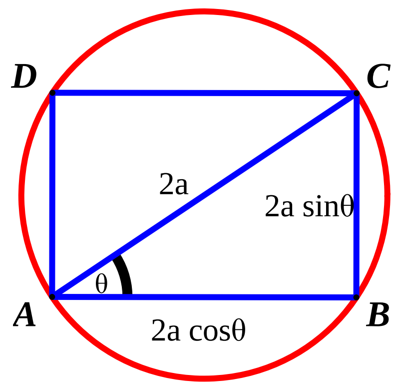
3. If the sum of the lengths of the hypotenuse and a side of a right angled triangle is given, show the area of the triangle is maximum when the angle between them is $\frac{\pi}{3}$.



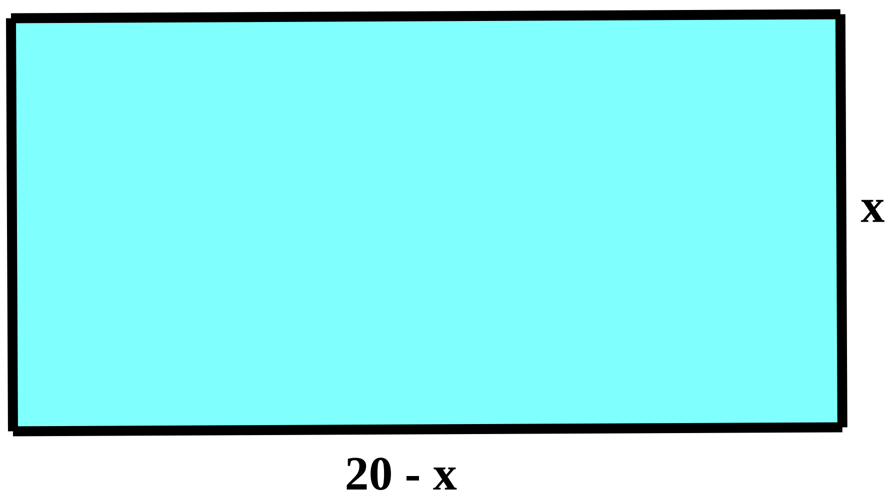
4. Show that of all the rectangles inscribed in a given fixed circle, the square has the maximum area.



5. Show that the rectangle of maximum perimeter which can be inscribed in a circle of radius a is a square of side a $\sqrt{2}$.



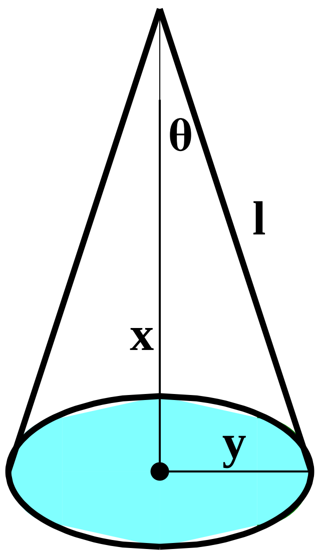
6. Of all the rectangles, each of which has perimeter 40 cm, find the one having the maximum area. Find the area also.



7. a) Show that the height of a right circular cylinder of maximum volume inscribed in a right circular cone of height h and radius r is $\frac{h}{3}$.

   b) Show that the volume of the greatest cylinder that can be inscribed in a cone of height h and semi-vertical angle α is $\frac{4}{27}\pi h^3 \tan^2 \alpha$.

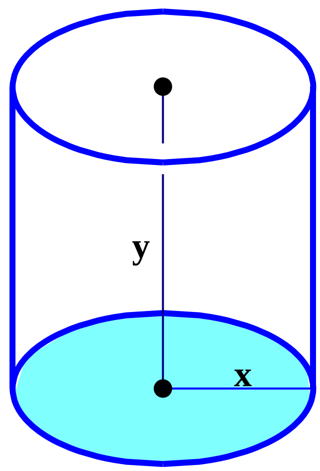
8. Show that the semi-vertical angle of the right circular cone of given slant height and maximum volume is $\tan^{-1}\sqrt{2}$ or $\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$.



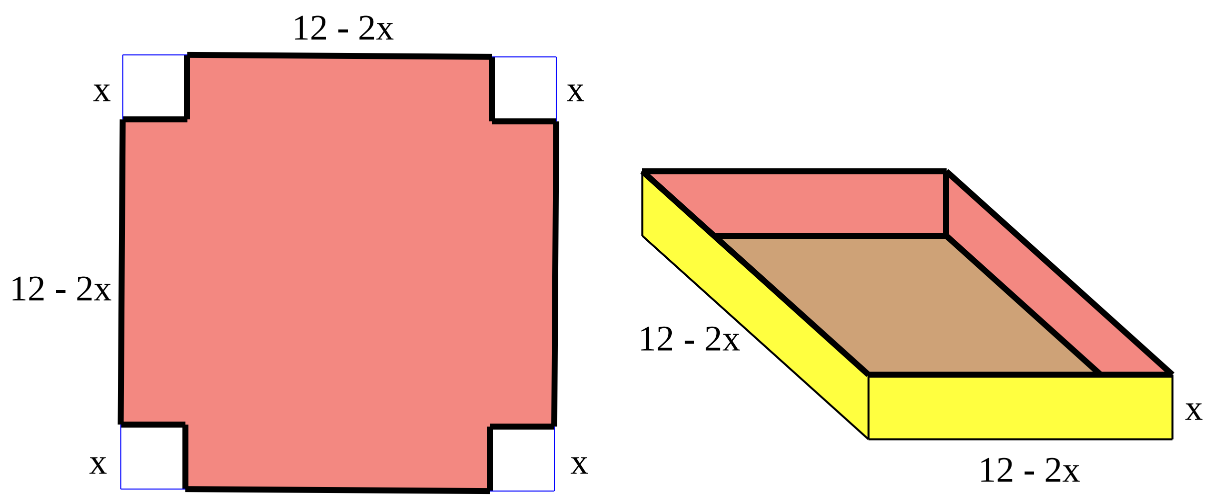
9. Find the altitude of a right circular cone of maximum curved surface which can be inscribed in a sphere of radius r.

10. a) Show that a closed right circular cylinder of given total surface area S and maximum volume V is such that its height h is equal to the diameter d of the base.

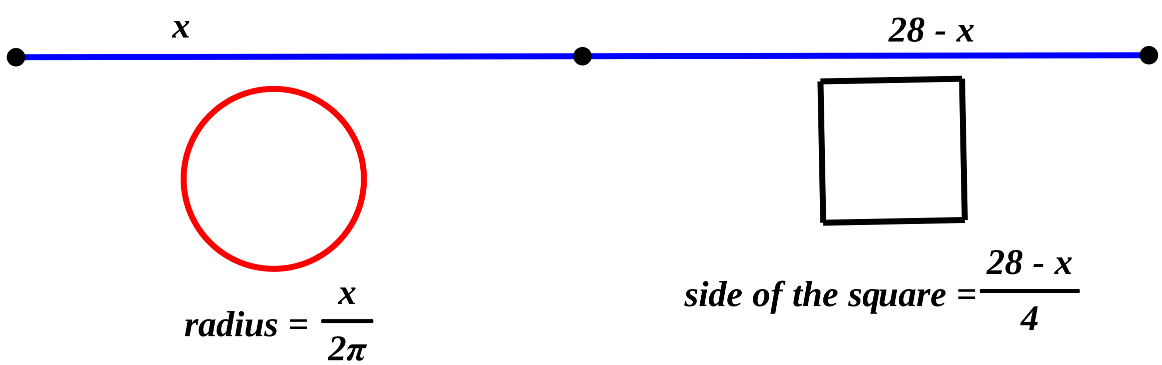
    b) Show that a right circular cylinder which is open at the top, and has a given total surface area, will have the greatest volume if its height is equal to the radius of its base.



11. A square piece of tin of side 12cm is to be made into a box without top by cutting a square from each corner and folding up the flaps to form a box. What should be the side of the square to be cut off so that the volume of the box is maximum? Also, find this maximum volume.



12. A wire of length 28 meters is to be cut into two pieces; one of the pieces is to be made into a square and the other into a circle. What should be the length of each piece so that the sum of the areas enclosed by them is minimum?



13. Divide the number 4 into two positive numbers such that the sum of the square of one and the cube of the other is a minimum.

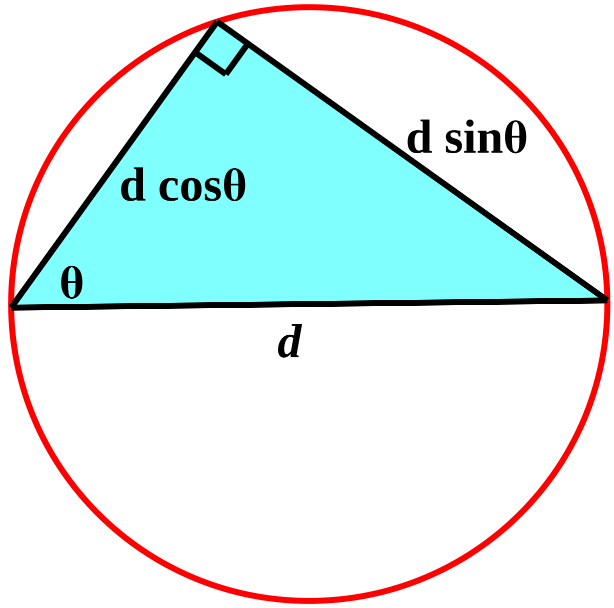
14. Find two numbers whose sum is 20 and sum of whose squares is minimum.

15. Find two such positive numbers whose sum is 16 and the sum of whose cubes is minimum.

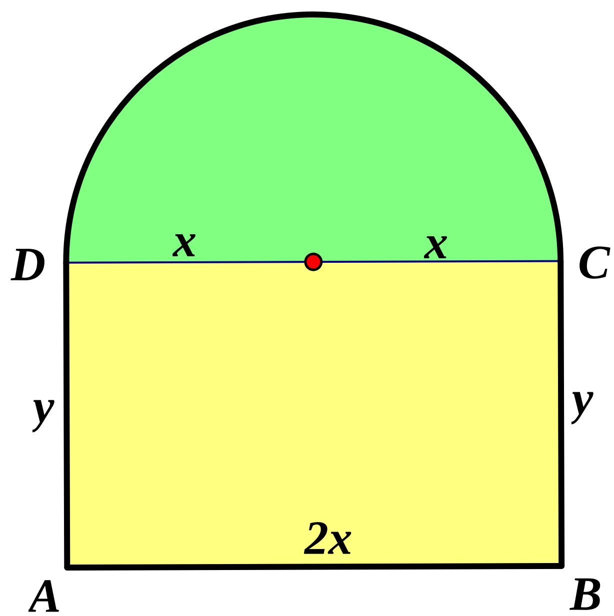
16. a) Show that of all the right triangles inscribed in a circle, the triangle with maximum perimeter is isosceles.

b) AB is a diameter of a circle and C is any point on the circle. Show that the area of

ΔABC is maximum, when it is isosceles.



17. a) A figure consists of a semi-circle with a rectangle on its diameter. Given the perimeter of the figure, find its dimensions in order that the area may be maximum.

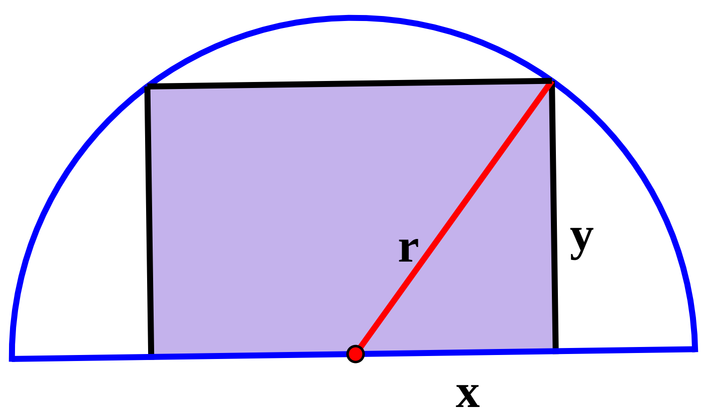


Hint: Assume the given perimeter to be P.

b) A window is in the form of a rectangle above which there is a semi-circle. If the perimeter of the window is p cm, show that the window will allow the maximum possible light only when the radius of the semi-circle is $\frac{p}{\pi + 4}$ cm.

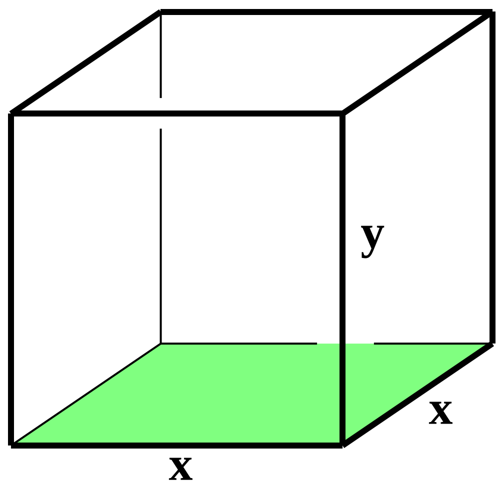
c) A figure consists of a semicircle with a rectangle on its diameter. Given that the perimeter of the figure is 20 feet, find the dimensions in order that its area may be maximum.

18. A rectangle is inscribed in a semi-circle of radius r, with one of its sides on the diameter of the semi-circle. Find the dimensions of the rectangle so that its area is maximum. Find also this area.



19. Find the volume of the largest cone that can be inscribed in a sphere of radius r.

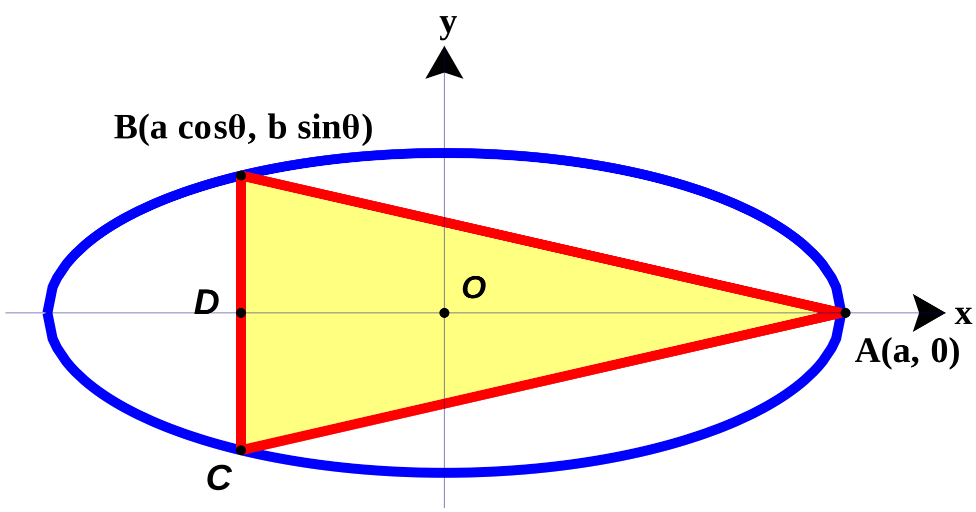
20. An open box with square base is to be made out of a given quantity of sheet of area a^2. Show that the maximum volume of the box is $\frac{a^3}{6\sqrt{3}}$.



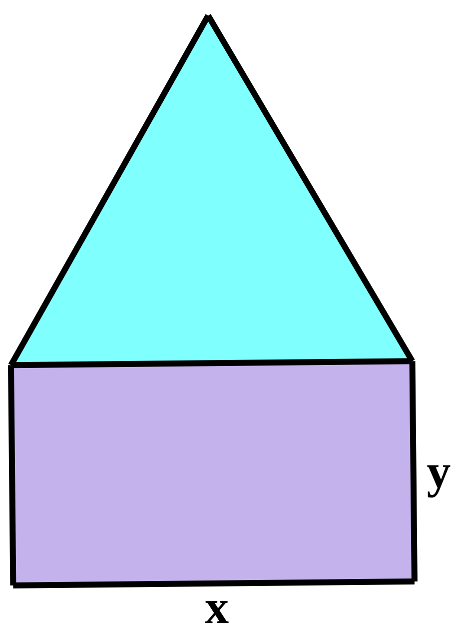
21. Show that the height of a cylinder of maximum volume which can be inscribed in a sphere of radius R is $\frac{2R}{\sqrt{3}}$. Also find the maximum volume.

22. a) Find the area of the greatest isosceles triangle that can be inscribed in a given ellipse with its vertex coinciding with one extremity of the major axis.

b) Find the maximum area of the isosceles triangle inscribed in the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with its vertex at one end of the major axis.



23. A rectangular window is surmounted by an equilateral triangle. Given that the perimeter is 16 m, find the width of the window so that maximum amount of light may enter.



24. Show that the height ‘h’ of a right circular cylinder of maximum total surface area, including the two ends, that can be inscribed in a sphere of radius ‘r’ is given by $h^2 = 2r^2\left(1-\frac{1}{\sqrt{5}}\right)$.

25. Show that the semi vertical angle of a right circular cone of given (total) surface area and maximum volume is $\sin^{-1}\left(\frac{1}{3}\right)$.

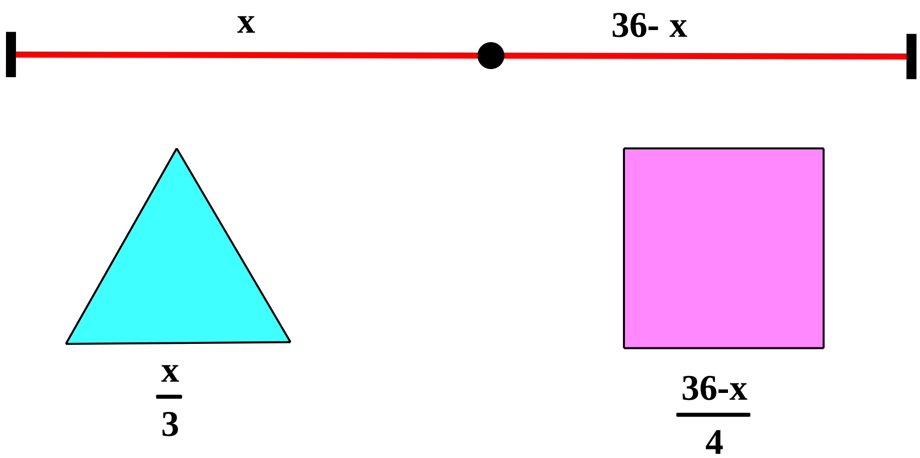
26. A rectangular sheet of tin $45\,\text{cm}\times 24\,\text{cm}$ is to be made into a box without top, by cutting off squares from each corner and folding up flaps. What should be the side of the square to be cut off from each corner so that the volume of the box is maximum?

27. a) Given the sum of the perimeters of a circle and a square, show that the sum of their areas is least when the diameter of the circle is equal to the side of the square.

b) The sum of the perimeter of a circle and square is K, where K is some constant. Prove that the sum of their areas is least when the side of the square is double the radius of the circle.

28. A wire of length 36 cm is cut into two pieces. One of the pieces is turned in the form of a square and the other in the form of an equilateral triangle. Find the length of

each piece so that the sum of the areas of the two is minimum.



29. An open box with a square base is to be made out of a given iron sheet of area 27 sq.m. Show that the maximum volume of the box is 13.5 cu.m.

30. Show that the maximum volume of the cylinder which can be inscribed in a sphere of radius $5\sqrt{3}$ cm is 500π cm^3.

31. Show that the height of the cone of maximum volume that can be inscribed in a sphere of radius 12 cm is 16 cm.

32. Prove that the volume of the largest cone that can be inscribed in a sphere of radius R is $\frac{8}{27}$ of the volume of the sphere.

33. A manufacturer can sell x items at a price of ₹$\left(5-\frac{x}{100}\right)$ each. The cost price of x items is ₹.$\left(\frac{x}{5}+500\right)$. Find the number of items he should sell to earn maximum profit.

34. x units of a given product can be manufactured at the total cost of ₹$\left(\frac{x^2}{100}+100x+40\right)$. The selling price per unit is ₹$\left(200-\frac{x}{400}\right)$. Find the production level at which the profit is maximum. Calculate the selling price per unit and the total profit at this level.

35. The cost function of a product of a firm per day for x units is given by

$C(x)=3000+271x+\frac{x^3}{6}$, whereas the revenue function is given by

$R(x)=3000+1000x-\frac{x^3}{6}$; 0 < x < 30.

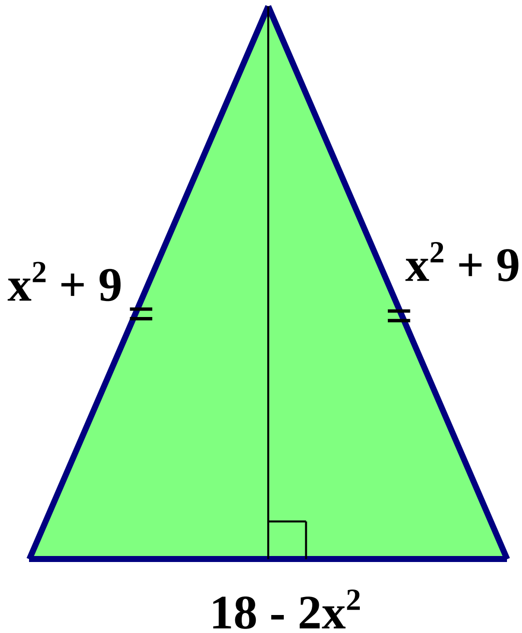
Calculate

a) the number of units that maximizes the profit; and

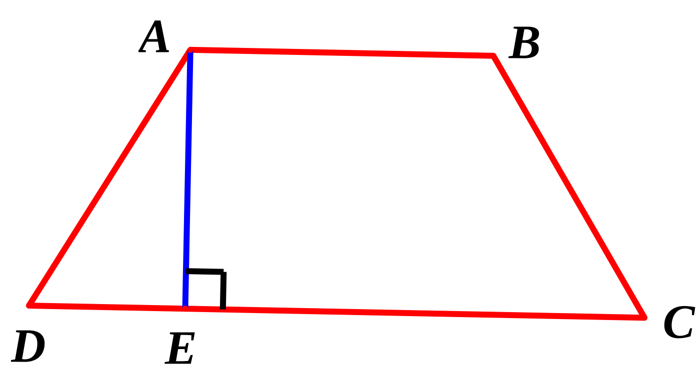
b) the profit per unit when the maximum profit level has been achieved.

36. The lengths of the sides of an isosceles triangle are $9+x^2, 9+x^2$ and $18-2x^2$ units. Calculate the area of the triangle in terms of x and find the value of x which makes

the area maximum.

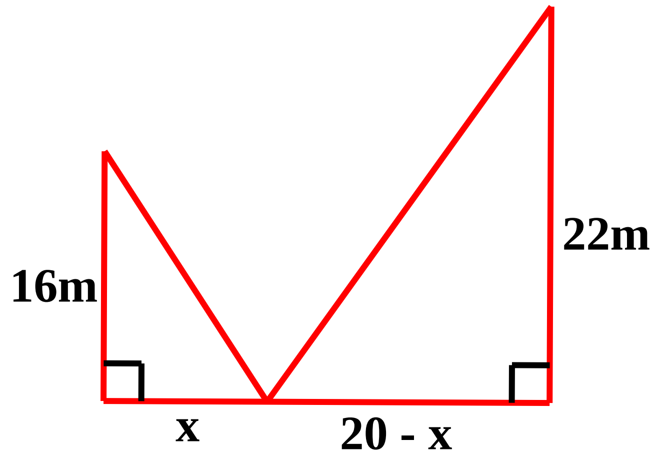


37. If the length of three sides of a trapezium other than the base is 10 cm each, find the area of the trapezium, when it is maximum.



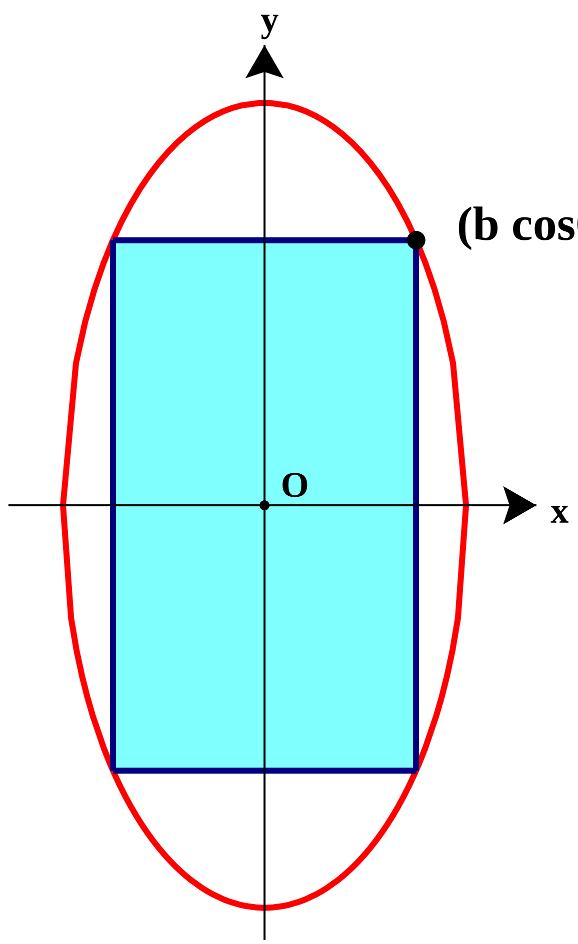
38. Two poles of heights 16 m and 22 m stand vertically on the ground 20 m apart. Find a point on the ground, in between the poles, such that the sum of the squares of the

distances of this point from the tops of the poles is minimum.

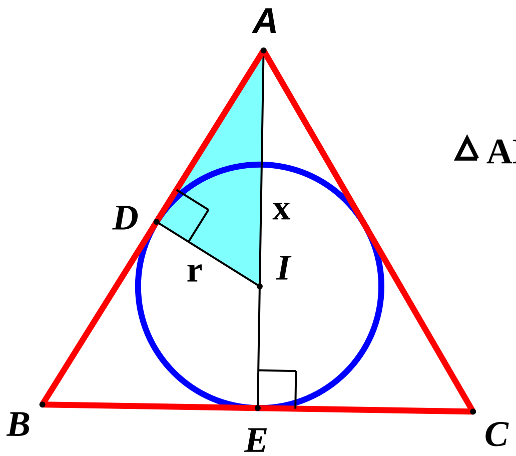


39. Prove that the radius of the right circular cylinder of greatest curved surface area which can be inscribed in a given cone is half of that of the cone.

40. Find the area of the greatest rectangle that can be inscribed in an ellipse $\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$



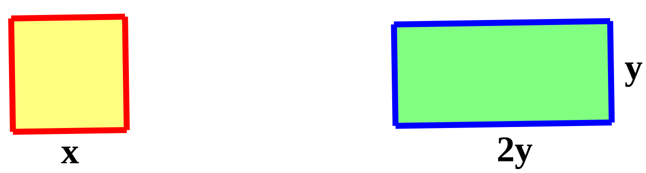
41. Prove that the least perimeter of an isosceles triangle in which a circle of radius r can be inscribed is $6\sqrt{3}r$.



Δ ADI ~ Δ AEB

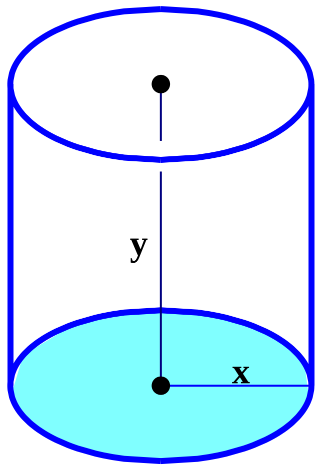
42. Find the maximum and minimum values of $f(x) = \sec x + \log\cos^2 x$, $0 < x < 2\pi$

43. A wire of length 34 m is to be cut into two pieces. One of the pieces is to be made into a square and the other into a rectangle whose length is twice its breadth. What should be the lengths of two pieces, so that the combined area of the square and the rectangle is minimum?



PROBLEMS DEFINED ON OPEN UNBOUNDED INTERVALS

44. A closed right circular cylinder has volume 2156 cubic units. What should be the radius of its base so that its total surface area may be minimum? Use $\pi = \frac{22}{7}$

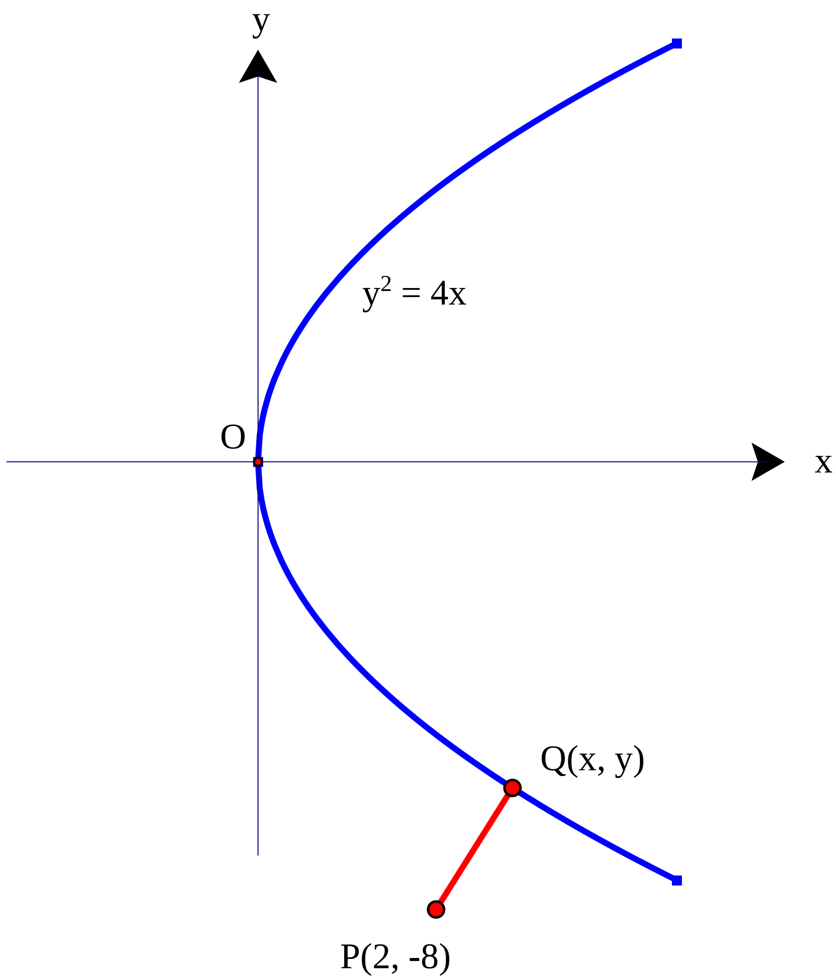


45. Find the height and diameter of a closed right circular cylinder of volume $128\pi \text{ cm}^3$ and minimum total surface area.

46. Show that a right-circular cylinder of given volume, open at the top, has minimum total surface area, provided its height is equal to the radius of the base.

47. Show that the surface area of a closed cuboid with square base and given volume is minimum when it is a cube.

48. a) Find the point on the curve $y^2 = 4x$ which is nearest to the point (2, -8).



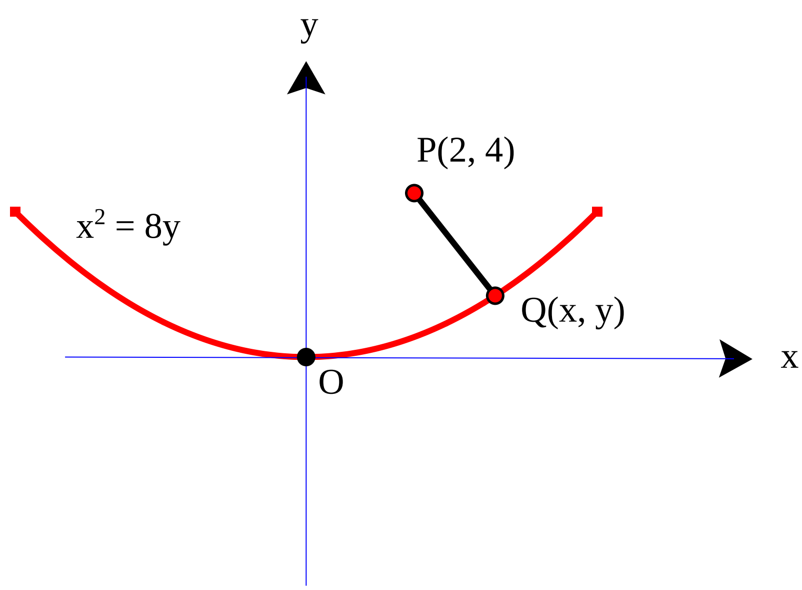
b) Find the point on the curve $y^2 = 2x$ which is nearest to the point (1, –4).

c) Find a point on the curve $y^2 = 2x$ which is nearest to the point (1, 4).

d) Find the point on the curve $y^2 = 4x$ which is nearest to the point (2, 1).

e) Find the point on the curve $x^2 = 4y$ which is nearest to the point (-1, 2).

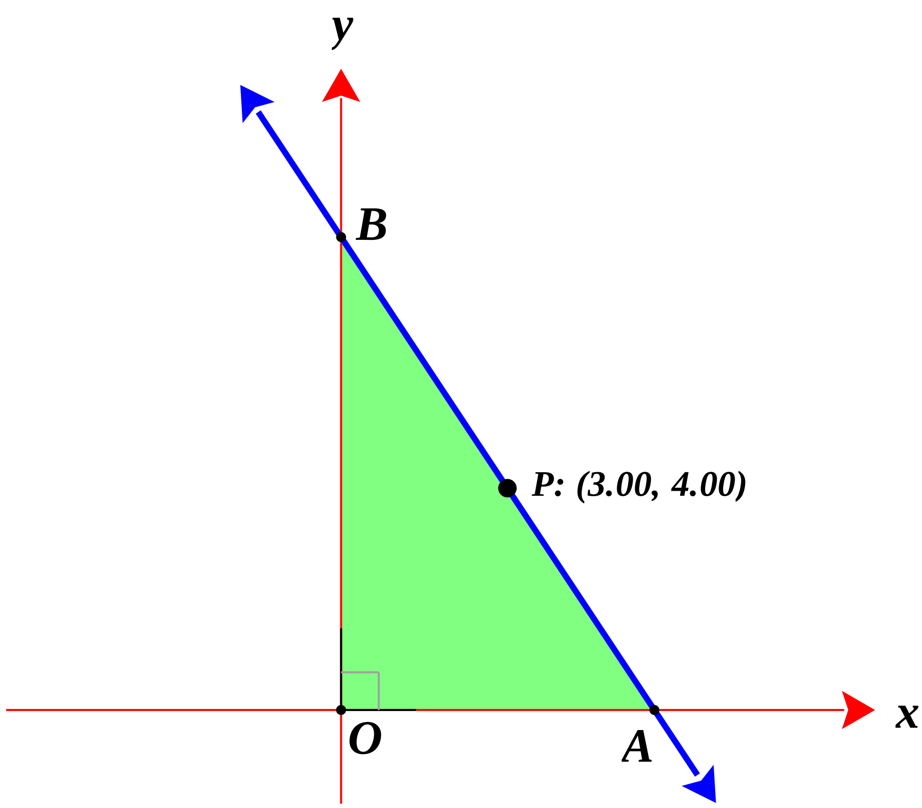
f) Find the point on the curve $x^2 = 8y$ which is nearest to the point (2, 4).



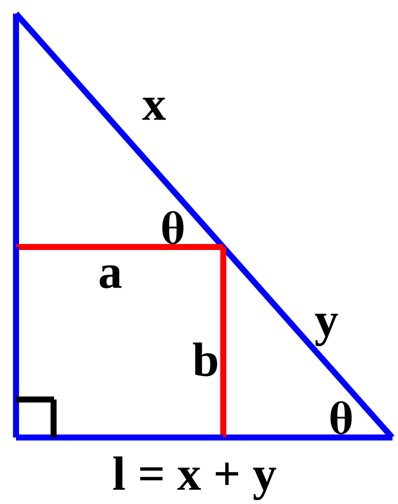
49. Find the radius of a closed right circular cylinder of volume 100 cubic centimeters which has the minimum total surface area.

50. An open tank with a square base and vertical sides is to be constructed from a metal sheet so as to hold a given quantity of water. Show that the cost of the material will be least when the depth of the tank is half of its width.

51. Find the equation of the line through the point (3, 4) which cuts from the first quadrant of a triangle of minimum area.



52. A point on the hypotenuse of a triangle is at distances a and b from the sides. Show that the minimum length of the hypotenuse is $\left(a^{\frac{2}{3}}+b^{\frac{2}{3}}\right)^{\frac{3}{2}}$.



53. A tank with rectangular base and rectangular sides, open at the top is to be constructed so that its depth is 2 m and volume is 8 cubic meters. If building of tank costs ₹ 70 per square meter for the base and ₹ 45 per square meter for sides, what is the cost of least expensive tank?

54. A square tank of capacity 250 cubic meters has to be dug out. The cost of the land is ₹50 per square meter. The cost of digging increases with depth and for the whole tank it is ₹ $400\times h^2$, where h meters is the depth of the tank. What should be the dimensions of the tank so that the cost is minimum?

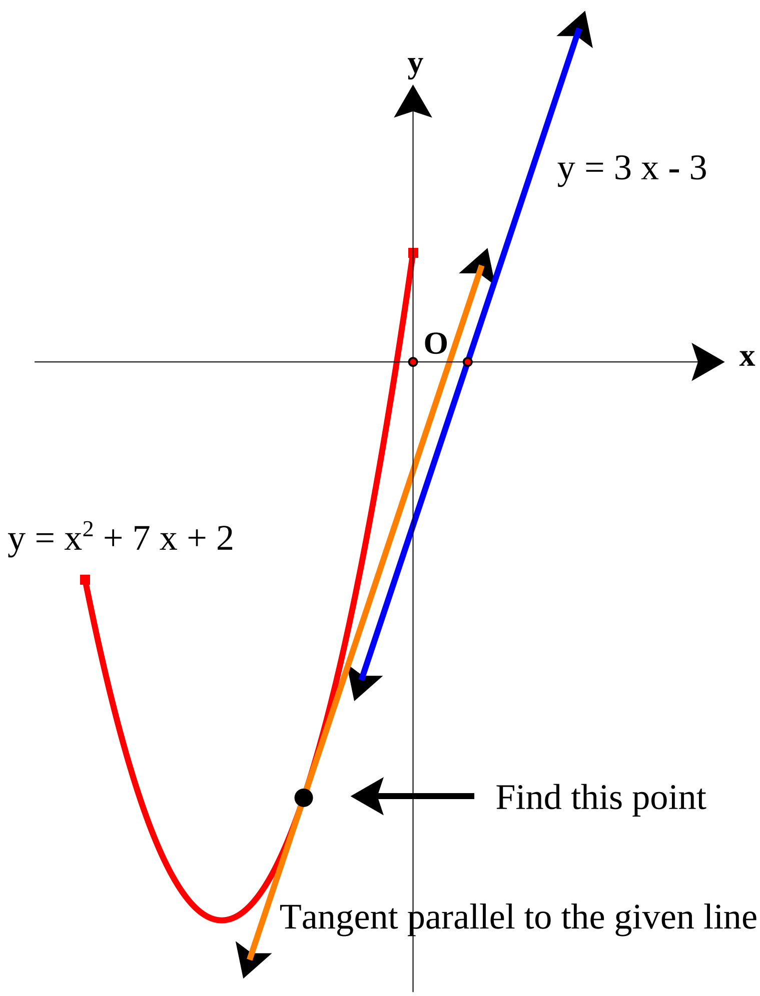
55. Show that the right circular cone of least curved surface and given volume has an altitude equal to $\sqrt{2}$ times the radius of the base.

56. Show that of all the rectangles of given area, the square has the smallest perimeter.

57. Show that the height of the closed right circular cylinder, of given volume and minimum total surface area, is equal to its diameter.

58. Find the minimum value of $(ax+by)$, where $xy=c^2$.

59. Find the coordinates of a point of the parabola $y = x^2 + 7x + 2$ which is closest to the straight line $y = 3x - 3$.

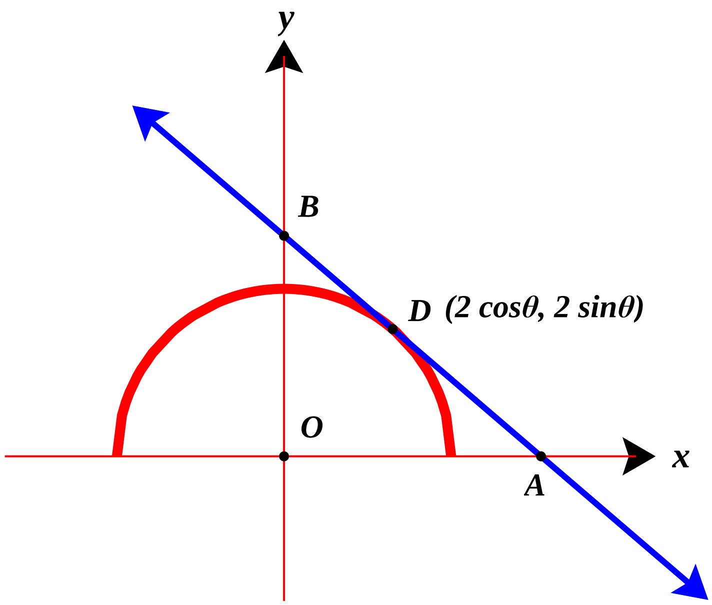


60. Find a point P on the curve $y^2 = 4ax$ which is nearest to the point (11a, 0).

61. A tank with rectangular base and rectangular sides open at the top is to be constructed so that its depth is 3 m and volume is 75 cubic meters. If building of tank costs

₹100 per square meters for the base and ₹50 per square meters for the sides, find the cost of least expensive tank.

62. Tangent to the circle $x^2 + y^2 = 4$ at any point on it in the first quadrant makes intercepts OA and OB on x and y axes respectively, O being the center of the circle. Find the minimum value of (OA + OB).



63. Find the point on the curve $y = \frac{x}{1+x^2}$, where the tangent to the curve has the greatest slope.

64. The sum of surface areas of a sphere and a cuboid with sides $\frac{x}{3}, x, 2x$ is constant. Show that the sum of their volumes is minimum if x is equal to three times the radius of sphere.

65. If the function $f(x)=2x^3-9mx^2+12m^2x+1$ where m > 0 attains its maximum and minimum at p and q respectively such that $p^2=q$, then find the value of m.

ANSWERS

1) $x=y=\frac{l}{\sqrt{2}}$, where l is the hypotenuse. Maximum area $\frac{l^2}{4}$.

2) $x=y=\frac{5}{\sqrt{2}}$, Area = $\frac{25}{4}$ square units.

3) $x=\frac{l}{3}, y=\frac{2l}{3}$, where l is the sum of the lengths of base and hypotenuse.

4) AB = $d\cos\frac{\pi}{4}$, BC = $d\sin\frac{\pi}{4}$ where d is the diameter of the circle.

5) Proof. AB = $2a\cos\frac{\pi}{4}$, BC = $2a\sin\frac{\pi}{4}$

6) Squares of side 10 cm. Maximum area = 100 square cm.

7) a) and b) are Proofs

8) Proof

9) Altitude = $\frac{4}{3}\times(Radius\ of\ the\ sphere)$

10) Proof

11) Squares of side 2 cm must be cut from 4 corners. Maximum volume = 128 cm^3.

12) Length of the wire made into circle $= \frac{28\pi}{4+\pi}$ meters.

Length of the wire made into square $= \frac{112}{4+\pi}$ meters

13) $\frac{8}{3}$ and $\frac{4}{3}$

14) 10 and 10

15) 8, 8

16) Proof

17) a) Radius $= \frac{P}{4+\pi}$, width $= \frac{P}{4+\pi}$, Length $= \frac{2P}{4+\pi}$

b) Proof

c) Radius $= \frac{20}{4+\pi}$ feet, width $= \frac{20}{4+\pi}$ feet, Length $= \frac{40}{4+\pi}$ feet

18) Length = $\sqrt{2}r$, breadth = $\frac{r}{\sqrt{2}}$, area = r^2

19) Maximum volume = $\frac{8}{27}\left(\frac{4}{3}\pi r^3\right)$

20) Maximum volume $\frac{a^3}{6\sqrt{3}}$.

21) Height = $\frac{2R}{\sqrt{3}}$. Maximum volume = $\frac{4\pi R^3}{3\sqrt{3}}$

22) a) Maximum area $= \frac{3\sqrt{3}}{4}ab$ b)same as a)

23) Width = $\frac{16}{6-\sqrt{3}}$ m, Height = $\frac{24-8\sqrt{3}}{6-\sqrt{3}}$ m

24) $h^2 = 2r^2\left(1-\frac{1}{\sqrt{5}}\right)$.

25) Semi vertical angle = $\sin^{-1}\left(\frac{1}{3}\right)$

26) 5 cm

27) a) and b) are proofs.

28) Square = $\frac{144\sqrt{3}}{9+4\sqrt{3}}$ m, Triangle = $\frac{324}{9+4\sqrt{3}}$ m

29) Maximum volume 13.5 cu.m.

30) Maximum volume 500 $\pi \; cm^3$.

31) 16 cm.

32) Proof

33) 240 items

34) 4000 items, ₹190, Total profit = ₹199960

35) 27 items, ₹486/unit

36) $x = \sqrt{3}$. Maximum area $36\sqrt{3}\, cm^2$

37) Maximum area $75\sqrt{3}\, cm^2$

38) At midpoint i.e., 10 m from the foot of each pole.

39) Proof

40) *2ab*

41) Proof

42) Maximum value is 1 at x = π, Minimum value = $2-\log 4$ at $\frac{\pi}{3}, \frac{5\pi}{3}$

43) 16 m is to be made into square and 18 m is to be made into rectangle.

44) 7 units

45) Diameter = height = 8cm

46) Proof

47) Proof

48) a) (4, –4) b) (2, –2) c) (2, 2) d) (1, 2).

e) (-2, 1) f) (4, 2).

49) $\sqrt[3]{\frac{50}{\pi}}$ cm

50) Proof

51) $\frac{x}{6}+\frac{y}{8}=1$

52) Proof

53) Least cost is ₹1000

54) Side of the square tank = 10 meters, depth of the tank = 2.5 m

55) Proof

56) $x = y = \sqrt{A}$, where A is the area of rectangle.

57) Proof

58) $2c\sqrt{ab}$

59) (–2, –8)

60) (9a, 6a) , (9a, –6a)

61) ₹5,500

62) $4\sqrt{2}$

63) (0, 0)

64) Proof

65) m = 2

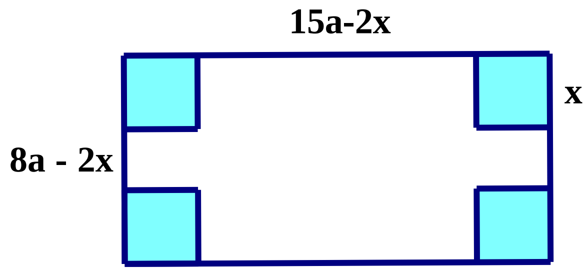
JEE QUESTIONS

MAXIMA AND MINIMA

1. **JEE (Advanced) 2015(I) – 44:** A cylindrical container is to be made from certain solid material with the following constraints: It has a fixed inner volume of V mm^3, has a 2 mm thick solid wall and is open at the top. The bottom of the container is a solid circular disc of thickness 2 mm and is of radius equal to the outer radius of the container.

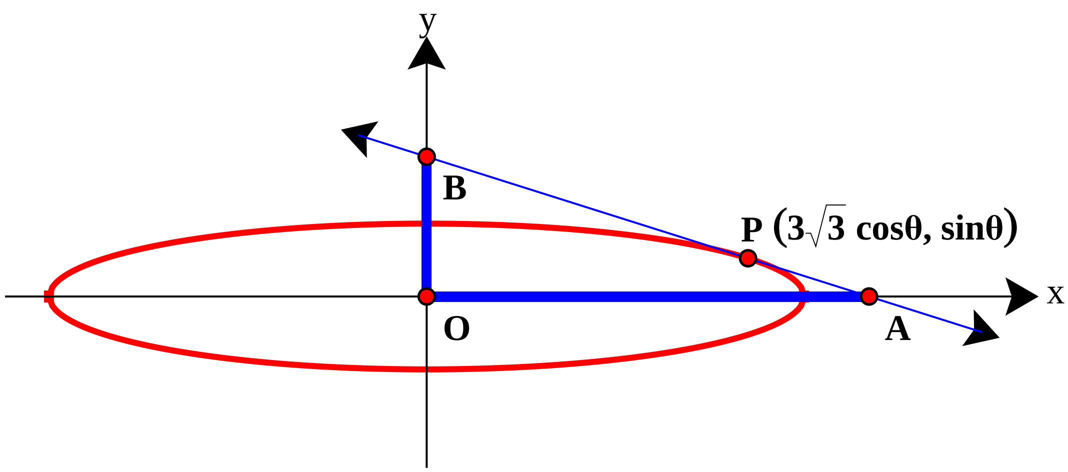
   If the volume of the material used to make the container is minimum when the inner radius of the container is 10 mm, then the value of $\frac{V}{250\pi}$ is ________

2. **JEE (Advanced)2013(I) – 52:** A rectangular sheet of fixed perimeter with sides having their lengths in the ratio 8 : 15 is converted into an open rectangular box by folding after removing squares of equal area from all four corners. If the total area of removed squares is 100, the resulting box has maximum volume. Then the lengths of the sides of the rectangular sheet are



A) 24 B) 32 C) 45 D) 60

3. Tangent is drawn to the ellipse $\frac{x^2}{27}+y^2=1$ at $\left(3\sqrt{3}\cos\theta, \sin\theta\right)$ where $\theta\in\left(0, \frac{\pi}{2}\right)$. Then the value of θ such that the sum intercepts between the axes made by this tangent is minimum, is



(A) $\frac{\pi}{3}$ (B) $\frac{\pi}{6}$

(C) $\frac{\pi}{8}$ (D) $\frac{\pi}{4}$

4. A tangent drawn at a variable point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ meets the coordinate axes at P and Q. The minimum area of the triangle formed by the tangent and the

coordinate axes is

(A) ab (B) $\frac{a^2+b^2}{2}$ (C) $\frac{(a+b)^2}{2}$ (D) $\frac{a^2+ab+b^2}{3}$

5. **AIEEE – 2009 (85) :** The shortest distance between the line $y-x=1$ and the curve $x=y^2$ is

A) $\frac{3\sqrt{2}}{8}$ B) $\frac{2\sqrt{3}}{8}$ C) $\frac{3\sqrt{2}}{5}$ D) $\frac{\sqrt{3}}{4}$

6. **AIEEE – 2003(58):** If the function $f(x)=2x^3-9ax^2+12a^2x+1$ where a > 0 attains its maximum and minimum at p and q respectively such that $p^2=q$, then 'a' equals *(Note: same question appeared in CBSE exam 2015)*

A) $\frac{1}{2}$ B) 3 C) 1 D) 2

HINTS AND ANSWERS

1. Answer = 4. Hint: Let $\pi r^2 h = V$ (auxiliary equation). Minimize S = Expression for metal used (volme of solid disc at the bottom, metal used for the pipe which is placed on the disc. $\frac{ds}{dr}=0$ when r = 10 mm.

2. A) C). Volume V of the box is maximum when x = 5.

3. B) Write the equation of the tangent at the given point. Use $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$. Minimize S = OA + OB.

4. A) Take the equation of the tangent at any point as $\frac{x\cos\theta x}{a}+\frac{y\sin\theta}{b}=1$.

5. A). Find the point on the curve where the tangent has slope =1.

6. D) Find f’(x) and equate it to zero. When x = p, f’’(x) < 0.